Simulation of Rocket

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1 Introduction

My project is to simulate the ascending and descending process of a rocket. The simulation can be seen as an expectation of how the rocket will behave given specifications, and give an idea of how the engineers should design the rocket. For simplicity, I simulation the rocket in one dimension (only consider about its altitude and vertical velocity). This allows the simulation to reflect the most dominant behavior of the rocket during the processes.

2 Equations

2.1 The Rocket Equation

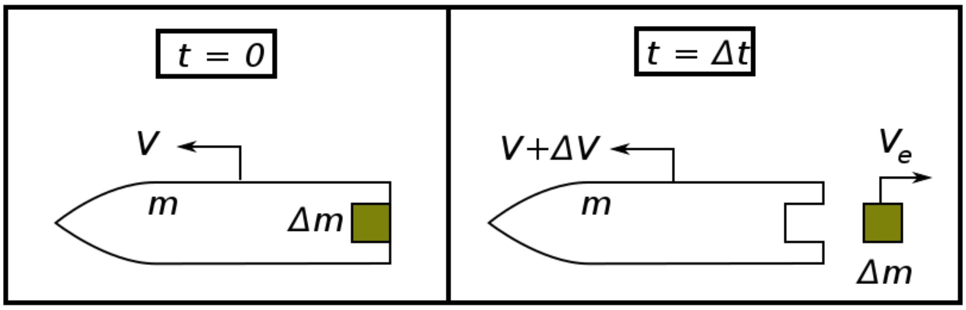

Figure 1: Generalization of a rocket-like system

We consider the system in figure 1, which is a generalization of rockets. The main idea of the system is that the system uses energy to accelerate part of its mass to the opposite direction in order to gain velocity by conservation of momentum. In our case, the ejected mass is the burnt fuel.

First, we consider there are no external forces exerted on the rocket (will consider them later), and the mass of the rocket is $m(t)$, a function of time t. Let $\mathbf{U}(t)$ be the velocity of the rocket, and $\mathbf{U}_{\text{fuel}}(t)$ be the velocity of the ejected fuel relative to the rocket. Then, the velocity in the inertial frame of the ejected fuel is $\mathbf{U}(t)+\mathbf{U}_{\text{fuel}}(t)$. Since there are no external forces exerted on the rocket (and the ejected fuel), the velocity of the fuel after ejection is constant. Therefore, at time t, the total momentum of the whole system is given by:

$$m(t)\mathbf{U}(t)+\int_{-\infty}^{t}-\frac{dm}{dt}(\tilde{t})\left(\mathbf{U}(\tilde{t})+\mathbf{U}_{\text{fuel}}(\tilde{t})\right)d\tilde{t} \tag{1}$$

Because the total momentum is conserved, the formula above yields 0. Differentiate both sides of the equation we have:

$$0=m(t)\frac{d\mathbf{U}}{dt}(t)+\frac{dm}{dt}(t)\mathbf{U}(t)-\frac{dm}{dt}(t)\left(\mathbf{U}(t)+\mathbf{U}_{\text{fuel}}(t)\right) \tag{2}$$

$$m(t)\frac{d\mathbf{U}}{dt}(t)=\frac{dm}{dt}(t)\mathbf{U}_{\text{fuel}(t)} \tag{3}$$

With this partial derivation of the rocket equation, we have had an ODE describing the change in velocity related to the ejection of fuel.

2.2 Density of Air

Since the rocket will reach a huge velocity in the atmosphere, air drag cannot be ignored. In our simulation of air drag (which will be introduced in the next sub-section), we need the air density around the rocket. For simplicity, I directly used the standard formula for air density in troposphere, and will not give the derivation here. The air density ρ at altitude h is given by:

$$\rho = \frac{p_0 M}{R T_0} \left(1 - \frac{Lh}{T_0}\right)^{\frac{gM}{RL} - 1} \tag{4}$$

where:

- p_0 is sea level standard atmospheric pressure, 101325 Pa
- M is molar mass of dry air, 0.0289652 kg/mol
- R is ideal (universal) gas constant, 8.31446 J/(mol·K)
- T_0 is sea level standard temperature, 288.15 K
- L is temperature lapse rate, 0.0065 K/m
- g is the gravitational constant, 9.8 m/s^2

Despite the fact that the gravitational constant will change with the change of altitude, since we consider the behavior of the rocket near the surface of the earth, where the change in g is minor, we will use the constant at the surface of earth all the time. With this assumption, all terms except h are constant in the equation above. Hence, the air density around the rocket is solely depend on the altitude of it.

2.3 Air Drag

We simulate air drag with the assumption that the air is very thin. In other words, we think the air molecules are at rest when the rocket hits them. Though this is not true at sea level, we will still take it in avoidance to any fluid dynamics. Suppose the rocket has a velocity U, it is the same as if the rocket is at rest, and the air molecules move towards the rocket at a velocity of $-U$. Also, let each air molecule has mass m, and their number per unit volume be σ. Let the diameter of the rocket be $2R$. In the simulation, we consider two different shape of the top of the rocket, one is semi-sphere and the other is cone.

Semi-sphere

As shown in figure 2, given a point in the semi-sphere, and let θ be the angle between the radius vector to the point and to the top. Though the figure is drawn in 2-D, note that the semi-circle in the figure is actually a semi-sphere, so there are a circle of points that have the same θ. The circumference of the ring is $2\pi R \sin\theta$. When an air molecule hits a point with angle θ, the change in momentum is:

$$\Delta p = mU\cos 2\theta - (-mU) = 2mU\cos^2\theta \tag{5}$$

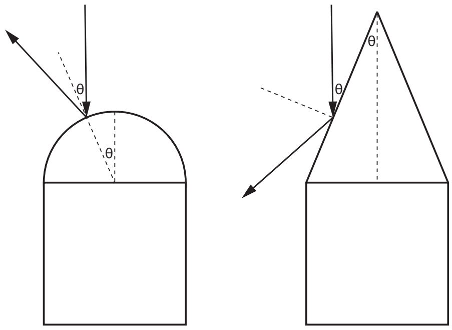

Figure 2: Left: air molecules collide with a semi-sphere. Right: air molecules collide with a cone

The area of the semi-sphere where points have angle $(\theta, \theta + d\theta)$ is given by:

$$dS = 2\pi R \sin\theta d(R\sin\theta) = 2\pi R^2 \sin\theta \cos\theta d\theta \tag{6}$$

We can get the number of collisions per unit time occur within $(\theta, \theta + d\theta)$ by multiplying equation (6) by $U\sigma$ and get

$$\text{\# of collisions} = U\sigma 2\pi R^2 \cos\theta \sin\theta d\theta \tag{7}$$

Since force is momentum per unit time, and any momentum gained by air molecules is lost by the rocket, the force resulted from air drag is the product of (5) and (7). Then, we integrate over $(0, \pi/2)$ and get:

$$F_{\text{drag}} = \int_0^{\pi/2} \left(2mU\cos^2\theta\right)\left(-2m\sigma U^2 2\pi R^2 \cos^2\theta \sin\theta\right) d\theta = -m\sigma U^2 \pi R^2 = -\rho U^2 \pi R^2 \tag{8}$$

where $\rho = m\sigma$ is the density of the air given in section 2.2.

Cone

Similarly to a semi-sphere, as shown in figure 2, we can analyze a cone-shape rocket in the same way. However, the change of momentum of a single air molecule is same everywhere on the cone, which is given by:

$$\Delta p = -mU\cos 2\theta - (-mU) = 2mU\sin^2\theta \tag{9}$$

Then, we only need to compute the side area of the cone, multiply it by (9) and $U\sigma$, we will get the force resulted from air drag is:

$$F_{\text{drag}} = -\pi \left(\frac{R}{\sin\theta}\right)^2 \sin\theta \left(2mU\sin^2\theta\right) U\sigma = -2\rho U^2 \pi R^2 \sin\theta \tag{10}$$

An interesting thing to notice is that when the angle of the cone is $\pi/6$, the rocket will have the same amount of air drag as that of a semi-sphere top.

2.4 System of ODEs

Combining the three components above, we can form a system of ODEs to describe the rocket. With the assumption that the rocket burns a constant amount mass of fuel R_m and ejects the exhaust at a constant velocity relative to the rocket $\mathbf{U}_{\text{fuel}}$, we have the following:

$$\frac{dh}{dt}(t) = \mathbf{U}(t) \tag{11}$$

$$\frac{d\mathbf{U}}{dt}(t) = \left(\frac{dm}{dt}(t)\,\mathbf{U}_{\text{fuel}}(t) - F_x(t)\right) \cdot \frac{1}{m(t)} - g(t) \tag{12}$$

$$\frac{dm}{dt}(t) = -R_m \tag{13}$$

Other alternatives to these ODEs will be introduced in the experiment section.

3 Numerical Methods

Since some numbers in the simulation is very large while others might be extremely small, it might easily have an unstable result if we use the naive Euler method. Therefore, I decided to use the more stable and accurate fourth-order Runge-Cotta method. The formula of fourth-order Runge-Cotta method is given by:

$$q_1 = f(t_k, y_k) \tag{14}$$

$$q_2 = f\left(t_k + \frac{h}{2}, y_k + \frac{h}{2}q_1\right) \tag{15}$$

$$q_3 = f\left(t_k + \frac{h}{2}, y_k + \frac{h}{2}q_2\right) \tag{16}$$

$$q_4 = f(t_k + h, y_k + hq_3) \tag{17}$$

$$y_{k+1} = y_k + \frac{h}{6}[q_1 + 2q_2 + 2q_3 + q_4] \tag{18}$$

$$\tag{19}$$

Basically, the method uses the idea of Simpson's rule for quadrature:

$$y(t+b) - y(t) = \int_t^{t+b} f(s)ds \approx \frac{h}{6}\left[f(t) + 4f\left(t + \frac{h}{2}\right) + f(t+b)\right] \tag{20}$$

Since the error in Simpson's rule is $O(h^5)$, Runge-Cotta method follows and also have the same rate of error. Another advantage of this method is that there are many built-in methods in Matlab or Scipy for it, making the coding part much easier.

4 Experiments

All the experiments in this section is simulated in 1-Dimension, where variables are positive when they point upward and negative when then point downward.

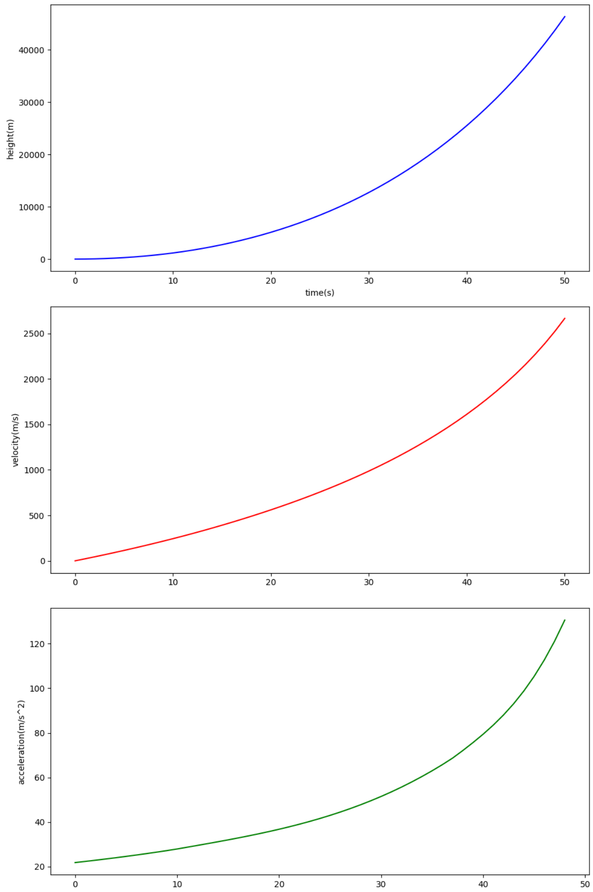

Figure 3: Graph of altitude, velocity, and acceleration in the setting without air drag

4.1 Without Air Drag

The first experiment is done under the assumption that there is no air drag. In other words, only the ejection and the gravity will change the velocity of the rocket. Figure 3 shows the result of the experiment. It can be seen from the graph that the acceleration grows (nearly) exponentially. This corresponds to the fact that the mass of the rocket decreases at a constant rate because of the burning of fuel. Which results in the velocity and the altitude grows even faster.

4.2 With Air Drag

Now we turn to a much more realistic setting where air drag is considered. Figure 4 shows the result of the experiment. It can be seen from the graphs, especially the graph of acceleration, that there are many phases during the flight of the rocket.

1. In the very beginning of the flight (about T+0s-T+5s), the total acceleration grows a little bit. This is because the velocity of the rocket is still very small, hence the air drag is minor. The acceleration due to ejection dominates at this stage, thus it grows a little bit due to the decrease in mass.

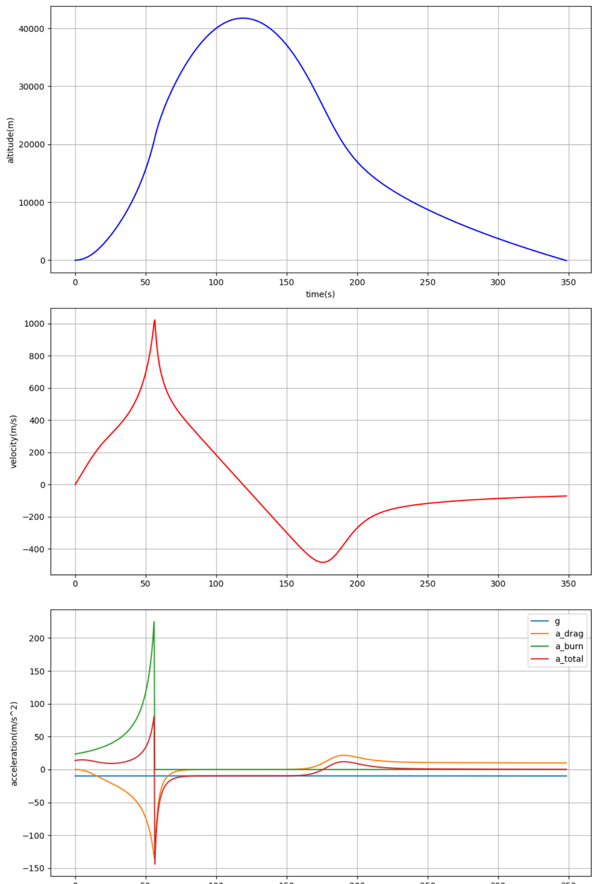

Figure 4: Graph of altitude, velocity, and acceleration in the setting with air drag

2. Then, the total acceleration decreases (about T+5s-T+30s). At this stage, the velocity is relatively large (approximately reaches the speed of sound), hence leading to a relatively larger air drag which dominates the total acceleration.

3. In the next phase (about T+30s-T+55s), the total velocity grows very sharp. Although the air drag is still increasing, the rate of increasing is not comparable to that of burning because the altitude of the rocket is relatively high, where air density is small.

4. At T+55s, we can see the acceleration due to burning goes to 0. It indicates that the rocket burns out of fuel. Only air drag and gravity act on the rocket after this time period.

5. From T+55s to T+75s, the velocity decreases very sharp. When it reaches about 200 m/s, the acceleration due to air drag goes nearly to 0. This makes sense because at this time the velocity isn't very large and the air is very thin. Therefore, gravity begins to dominate after T+75s

6. From T+75s-T+160s, gravity dominates the total velocity. Velocity decrease to negative in a constant rate. The rocket reaches the peak of its trajectory and begins to fall.

7. From T+160s to T+220s, the rocket falls back to the thick atmosphere. Since the absolute value of velocity is larger than the equilibrium, air drag grows and forces the absolute value of velocity to decrease (note that here air drag is positive and velocity is negative).

8. Afterwards (T+220s-T+350s), the rocket reaches the equilibrium state where the absolute value of air drag nearly equals that of gravity so the total acceleration converges to 0. The experiment ends when the rocket reaches the ground (altitude be 0 again).

This experiment clearly reflects many different phases during the flight of the rocket, the next to experiments are all little modifications to this one.

4.3 Different Angles of Cones

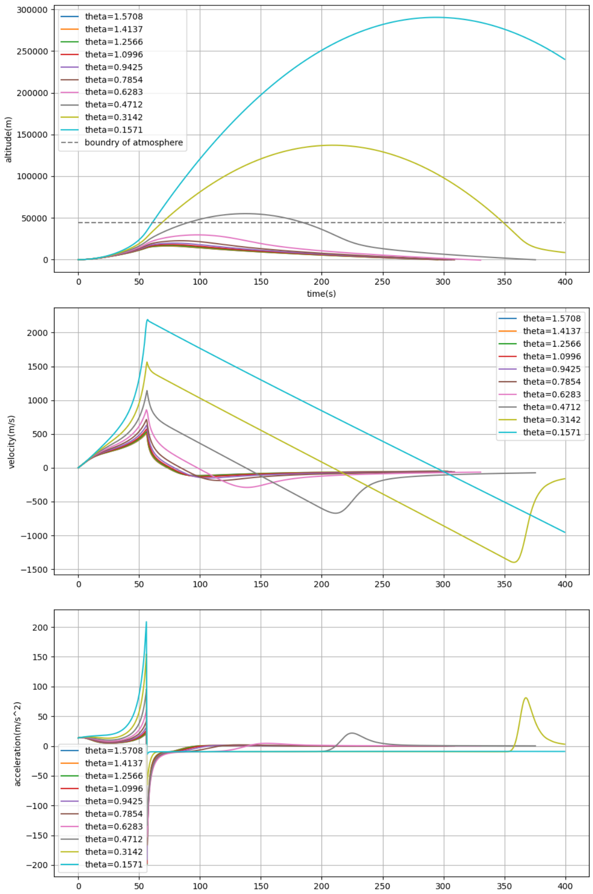

Figure 5: Graph of altitude, velocity, and acceleration in the setting with different angles of cones as the top of the rocket

As we explained in section 2.3, air drag would be different if the top of the rocket is a cone. In this experiment, we tried different shapes of cones. More specifically, we change the angle θ of the cone that is shown in figure 2. The result of the experiment is shown in figure 5.

We can see from the graph of acceleration that the smaller the angle is, the less air drag the rocket would have. When θ is less than 0.15π, the rocket can even goes out of the boundary of the atmosphere (where the air density defined by (4) is 0) There are some interesting behavior to notice:

- When the top of the rocket is flat ($\theta = \pi/2$), we don't see a peak of acceleration when the rocket falls back to the thick part of the atmosphere, which means the rocket reaches the equilibrium state when it begins to fall.
- When the rocket gets higher in the peak of its trajectory, it will suffer from larger air drag when it falls back to the thick part of the atmosphere.
- It can be inferred from (10) that when the cone gets infinitely long (when θ goes to the limit of 0), the rocket will have 0 air drag.

4.4 Landing of a Rocket

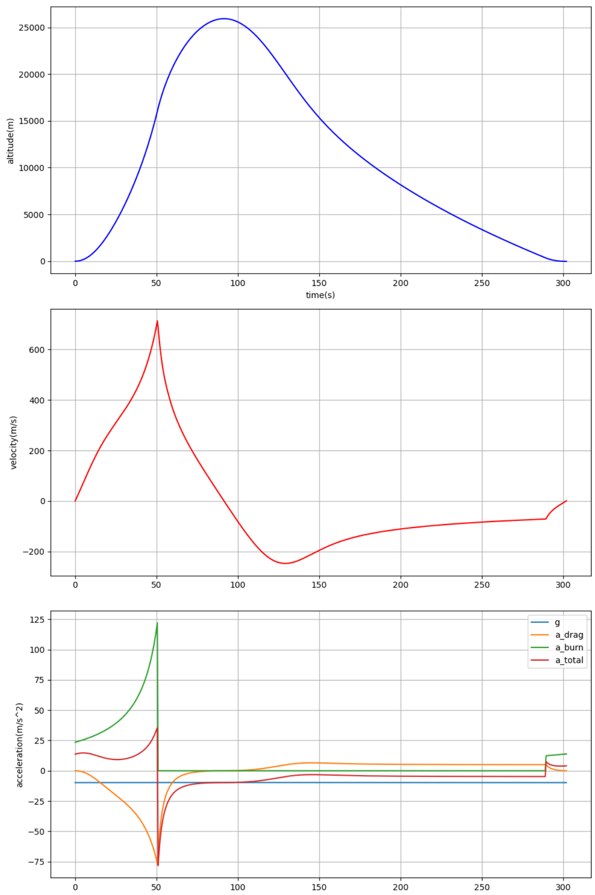

Figure 6: Graph of altitude, velocity, and acceleration in the setting to land the rocket

In this experiment, everything is the same except that we aim to land the rocket softly. In other words, we hope the velocity of the rocket gets very close to 0 when the altitude reaches 0. To do so, we need to save a portion of fuel during ascending stage (in this experiment, 10%), and burn the saved fuel during the descending stage. Since the rocket is much lighter in the final stage compared to the beginning, we need to throttle down the engine to have a more gentle landing burn. In this experiment, we let both the rate of fuel burning and the ejection velocity to be 30% of the original ones, which lead to an approximately 10% of burning acceleration in the starting of the landing burn compared to the time of engine shut down. The result of the experiment is shown in figure 6. It can be seen from the graph of velocity that it reaches very close to 0 and the rocket reaches the ground.

5 Summary and Conclusions

In general, by combining the rocked equation, the air density formula, and the air drag formula, I'm able to make the simulations straightforward and reasonable. With the help of Runge-Cotta method, I'm also able to solve the ODEs numerically in a relatively accurate sense. In the experiment, I focused on different behaviors of the rocket at different stages of the flight. These behaviors make great sense because they can be explained by the variance of different variables. In other words, the rocket behaves like what we would expected. Based on that, I explored a bit to find what would happen if we vary the top of the rocket or what if we want to land the rocket softly.

Acknowledgement

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References

[1] Greenbaum, Anne and Timothy P. Chartier. *Numerical Methods: Design, Analysis, and Computer Implementation of Algorithms*.

[2] Peskin, Charles. *Lectures on Orbits of Planets and Spacecraft*.

[3] Wikipedia. *Density of air*. Accessed https://en.wikipedia.org/wiki/Density_of_air

[4] Wikipedia. *Tsiolkovsky rocket equation*. Accessed https://en.wikipedia.org/wiki/Tsiolkovsky_rocket_equation

Appendix A Relavent links

Codes of this project can be viewed on GitHub at:

https://github.com/Gaaaavin/rocket-simulation